Int. J. Solids Structures Vol. 27, No. 5, pp. 639-646, 1991
Printed in Great Britain.

0020-7683/91 $3.00 + .00
© 1990 Pergamon Press plc

POST-BUCKLING ANALYSIS OF COMPOSITE DELAMINATED BEAMS

IZHAK SHEINMAN and MICHAEL SOFFER
Faculty of Civil Engineering, Technion—Israel Institute of Technology, Technion City, Haifa 32000, Israel

(Received 15 June 1989)

Abstract—A general one-dimensional geometrical non-linear model of a composite delaminated beam under arbitrary axial and transverse loading for any boundary conditions is derived for predicting the post-buckling behavior. The model comprises nonlinear equations based on the Von Karman kinematic approach, suitable for checking the effect of delamination and initial imperfection on the overall non-linear behavior. The differential equations, which incorporate the bending-stretching coupling effect, are solved by Newton's method using a finite-difference scheme. The delamination ratio parameter, bending-stretching coupling and the initial imperfection are examined by means of parameteric analysis.

1. INTRODUCTION

Application of advanced composite materials entails advanced analytical tools and insight into the response and failure characteristics of laminated composite structures. One mode of failure is the so-called delamination effect, which consists of interply fracture occurring during fabrication or under the action of service factors such as impact loading. It may significantly impair the load-carrying capacity of the laminate, mainly through local instability created in the vicinity of the affected spot. Such local instability (represented analytically by the first bifurcation point, given by a system of eigenvalues) does not necessarily imply the ultimate load, and usually the laminate is capable of carrying on in a post-buckling mode under higher loading.

The bifurcation point is discussed in recent works (Chai *et al.*, 1981; Yin *et al.*, 1984; Simitses *et al.*, 1985; Shirakumar and Whitcomb, 1985; Sheinman *et al.*, 1989) on the delamination problems of one-dimensional models. Yin (1986) and Chai *et al.* (1981) also deal with the post-buckling shape, but in terms of local delamination growth rather than of geometrical non-linear analysis. In most of the above studies, no allowance is made for the extremely important bending stretching coupling effect, which makes for drastic reduction of the overall non-linear behavior. It should be noted that even if the plies are initially arranged in a symmetric pattern (in which case there is no coupling), this symmetry is disturbed on delamination and coupling becomes possible.

The present study is an attempt to provide a suitable analysis for the non-linear behavior of a composite delaminated beam with initial imperfection under external loading, at an arbitrary stacking combination and orientation and under arbitrary boundary conditions. It is an extension of Sheinman and Adan's therory (1987) on delaminated beams, and of Sheinman *et al.*'s work (1989) on post-buckling behavior. The geometrical model is the one employed by Simitses *et al.* (1985) (also see Sheinman *et al.*, 1989), whereby a laminated composite beam of thickness t containing (at depth h below the top surface; see Fig. 1) a parallel plane crack extends over the lengthwise interval $l_2 = l_3$ and across the whole width. The crack, referred to as a "delamination", is assumed to exist before loading is applied and not to grow under it. The beam is subjected to arbitrary axial and transverse loading. The thickness-to-span (and to delamination length) is also assumed to be very small, so that the shear deformation can be neglected (see Sheinman and Adan, 1987). The delamination subdivides the beam into four regions, represented by equilibrium equations, continuity requirements at the crack tips, and boundary conditions at the ends. The non-linear differential equations are derived for the initial imperfection parameter of each region.

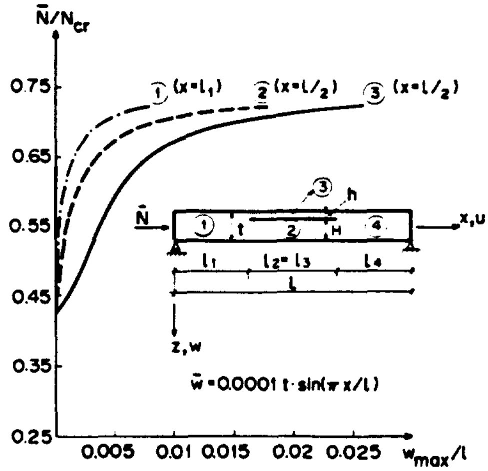

Fig. 1. Load-deflection curves for an isotropic simply-supported beam with positive initial imperfection for delamination length parameter $\alpha = l_1/l = 0.375$.

The solution procedure is based on reduction of the non-linear differential equations to a linear sequence by a modification of Newton's method, and conversion to an algebraic one by finite differences. A parametric study of the effect of delamination and initial imperfection on the non-linear behavior was carried out by applying the procedure to isotropic and anisotropic beams.

2. GOVERNING EQUATIONS

Kinematics

The Kirchhoff–Love hypothesis, whereby a normal to an undeformed surface remains unstretched after deformations, is adopted as the basic assumption. This leaves only two independent variables, namely the displacements in the axial (u) and normal (w) directions, represented by the x- and z-coordinates, respectively (see Fig. 1).

The kinematic relation for each region is:

$$
\begin{aligned}
U(x,z) &= u(x) - zw_{,x}(x) \\
W(x,z) &= w(x) \\
\varepsilon(x,z) &= \varepsilon^0(x) + z\chi(x)
\end{aligned}
\tag{1}
$$

where $\varepsilon^0(x)$ and $\chi(x)$ are respectively, the strain of the reference surface and change of curvature under deformation, which, associated with the displacement field and imperfection function $\bar{w}(x)$, can be written as:

$$
\begin{aligned}
\varepsilon^0(x) &= u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x}) \\
\chi(x) &= -w_{,xx}.
\end{aligned}
\tag{2}
$$

Constitutive equations

Under the classical laminate theory (i.e. for a single anisotropic equivalent layer) the force–strain relations can be written as:

$$
\begin{aligned}
N &= A_{11}\varepsilon^0 - B_{11}w_{,xx} \\
M &= B_{11}\varepsilon^0 - D_{11}w_{,xx}
\end{aligned}
\tag{3}
$$

where N and M are the axial force and bending moment, which are given by

$$(N, M) = \int_A S_{xx}(1, z)\, \mathrm{d}A. \tag{4}$$

S_{xx} is the Kirchhoff stress in the undeformed system, A, B, and D being the usual stiffness parameters employed in laminate theory:

$$(A_{11}, B_{11}, D_{11}) = b \int_{h_B}^{h_T} Q_{11}(1, z, z^2)\, \mathrm{d}z. \tag{5}$$

b is the beam width and Q_{11} the elastic stiffness transformed to the x direction.

The bending moment can alternatively be expressed as (see Whitney, 1987)

$$M = b_{11} N - d_{11} w_{,xx}. \tag{6}$$

The classical one-dimensional analysis involves the development of the laminated beam theory (Whitney, 1987):

$$\begin{aligned} b_{11} &= B_{11}/A_{11} \\ d_{11} &= D_{11} - B_{11}^2/A_{11}, \end{aligned} \tag{7}$$

while for behavior which is referred to as cylindrical bending (see Sheinman, 1989):

$$\begin{aligned} b_{11} &= ([A^{-1}][B])_{11} \\ d_{11} &= ([D] - [B][A^{-1}][B])_{11}. \end{aligned} \tag{8}$$

Equilibrium equations

By applying the variational principle, the following non-linear equations are obtained for a straight beam:

$$\begin{aligned} N_{,x} &= -p_x \\ M_{,xx} + N(w_{,xx} + \bar{w}_{,xx}) + p_x(w_{,x} + \bar{w}_{,x}) &= -p_z \end{aligned} \tag{9}$$

with the boundary conditions imposed on

$$\begin{aligned} &u \quad \text{or} \quad N \\ &w \quad \text{or} \quad M_{,x} + N(w_{,x} + \bar{w}_{,x}) \\ &w_{,x} \quad \text{or} \quad M. \end{aligned} \tag{10}$$

p_x and p_z are the external distributing loads in the axial and transverse directions, respectively.

Substituting eqns (3)–(6) in (9) and (10), we obtain the non-linear equilibrium equation in terms of displacements:

$$\begin{aligned} &A_{11}[u_{,xx} + \tfrac{1}{2}w_{,xx}(w_{,x} + 2\bar{w}_{,x}) + \tfrac{1}{2}w_{,x}(w_{,xx} + 2\bar{w}_{,xx})] - B_{11}w_{,xxx} = -p_x \\ &-d_{11}w_{,xxxx} + [A_{11}(u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x})) - B_{11}w_{,xx}]\cdot(w_{,xx} + \bar{w}_{,xx}) + p_x(w_{,x} + \bar{w}_{,x}) + b_{11}p_{x,x} = -p_z, \end{aligned} \tag{11}$$

the boundary conditions being:

$$\begin{aligned} &u \quad \text{or} \quad A_{11}[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + \bar{w}_{,x})] - B_{11}w_{,xx} \\ &w \quad \text{or} \quad -d_{11}w_{,xxx} + b_{11}p_x + [A_{11}(u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + \bar{w}_{,x})) \\ &\qquad\qquad - B_{11}w_{,xx}](w_{,x} + \bar{w}_{,x}) \\ &w_{,x} \quad \text{or} \quad B_{11}[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + \bar{w}_{,x})] - D_{11}w_{,xx}. \end{aligned} \tag{12}$$

Continuity requirements

The following continuity requirements are applied at the crack tip (see Simitses *et al.*, 1985; Sheinman *et al.*, 1989).

Kinematic continuity conditions

$$\begin{aligned}
{}^{(2)}u &= {}^{(g)}u - \frac{h}{2}\,{}^{(g)}w_{,x} \\
{}^{(3)}u &= {}^{(g)}u + \frac{h}{2}\,{}^{(g)}w_{,x} \\
{}^{(2)}w &= {}^{(3)}w \\
{}^{(3)}w &= {}^{(g)}w \\
{}^{(2)}w_{,x} &= {}^{(3)}w_{,x} \\
{}^{(3)}w_{,x} &= {}^{(g)}w_{,x}.
\end{aligned} \tag{13}$$

Continuity of moments and forces

$$\begin{aligned}
{}^{(g)}M - {}^{(2)}M - {}^{(3)}M + {}^{(3)}N\left(\frac{t-h}{2}\right) - {}^{(2)}N\left(\frac{t-H}{2}\right) &= 0 \\
-{}^{(g)}\hat{Q} + {}^{(2)}\hat{Q} + {}^{(3)}\hat{Q} &= 0 \\
-{}^{(g)}N + {}^{(2)}N + {}^{(3)}N &= 0
\end{aligned} \tag{14}$$

where the left-hand superscript (g) denotes the numerical designation of the region of the segment adjoining the crack tip: for the first crack tip $g = 1$ and for the second $g = 4$. The shear force $\hat{Q}$ is defined as

$$\hat{Q} = M_{,x} + N(w_{,x} + \bar{w}_{,x}). \tag{15}$$

The solution procedure is mainly the same as in Sheinman and Adan (1987), where a modification of Newton's method is employed for reducing the non-linear equations to a linear sequence, and a central finite difference scheme with fictitious points is used to convert the differential equations into algebraic ones. Finally, this algebraic set of equations is solved by a modification of Potter's method (Sheinman and Simitses, 1984).

3. NUMERICAL RESULTS AND DISCUSSION

For the procedure outlined above, a general computer program NADB (Non-linear Analysis of Delaminated Beams) was written, covering non-linear behavior of any delaminated composite beam with arbitrary stacking combination and orientation under arbitrary external loading and boundary conditions, as well as any geometrical imperfection.

Two cases of a simply-supported beam under axial compression were considered: (a) isotropic and (b) anisotropic.

(a) *Isotropic beam*

This example is reproduced from Sheinman and Adan (1987) and demonstrates the effect of the initial imperfection as well as of the delamination ratio parameter ($\alpha = l_3/l$). The data for this example are: length $l = 4.0$ m, width $b = 0.04$ m, thickness of laminate $t = 0.08$ m, thickness of delaminated upper layer $h = 0.01$ m ($H = 0.07$), modulus of elasticity $E = 2.1 \times 10^{11}$ N m^{-2} and Poisson's ratio $\nu = 0.3$. The initial imperfection was taken as $\bar{w}(x) = \delta \sin(\pi x/l)$. The non-linear behavior was first examined for a positive initial imperfection with small amplitude. In Fig. 1 the load–deflection curves of all regions

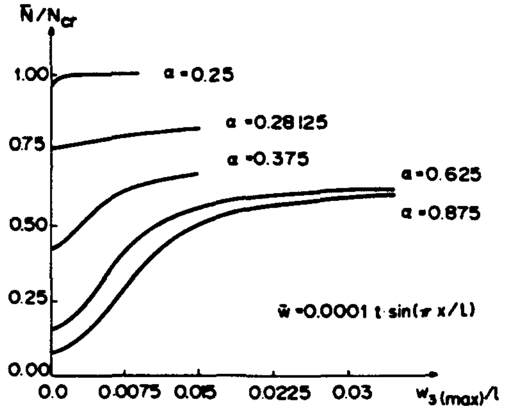

Fig. 2. Effect of delamination ratio ($\alpha = l_3/l$) on non-linear behavior of an isotropic simply-supported beam with positive initial imperfection.

(region 1 at $x = l_1$, regions 2 and 3 at $x = l_1 + l_2/2$) are plotted for $\alpha = l_3/l = 0.375$. One can see that at the load level of $N \approx 0.4N_{cr}$ (where N_{cr} is defined as the buckling load of the undelaminated beam), the delaminated upper layer (region 3) is buckled. Then the total stiffness of all regions is reduced, but the load capacity remains higher up to about $0.7N_{cr}$. In other words, the bifurcation point for delaminated beams is only an indication of the overall behavior, and does not represent the load capacity, and the post-buckling behavior should be considered. The effect of α is shown in Fig. 2, in which it is plotted versus deflection of the delaminated upper layer. It is seen that as α increases, the load capacity decreases. Above the local buckling load of the upper layer, the static scheme changes and the internal force in this region is reduced, as illustrated in Fig. 3. The explanation is that as the upper layer buckles, the eccentricity increases and so does the tensile force due to the moment, so that the compressive force decreases.

The local buckling load, represented by the first bifurcation point (in region 3), is given by $N_{3cr} = \beta\pi^2(EI)_3/l_3^2$, where $(EI)_3$ and l_3 are the stiffness and length of region 3 which buckles first. β is a parameter which depends on the end constraints of region 3, which are in turn a function of the imperfection amplitude. For a perfect beam the upper layer can be considered as a clamped–clamped beam with $\beta = 4$. In Fig. 4, β is plotted versus the imperfection amplitude for some values of α.

It should be noted here that the present methodology does not allow for the contact phenomenon. So, for some shape of the imperfection, local buckling cannot occur (when the delaminated upper layer buckles toward the lower layer). For this reason we also consider the behavior under a negative initial imperfection, where buckling of the upper layer is possible.

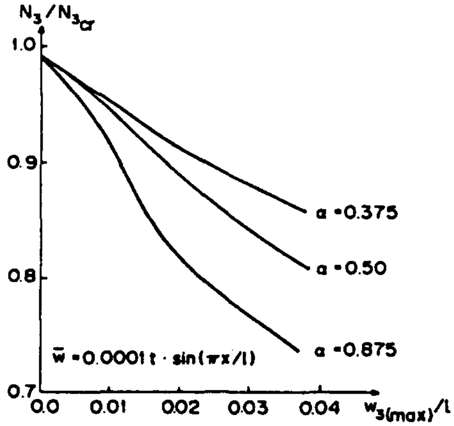

Fig. 3. Load–deflection curves of region 3 for different values of delamination ratio (α).

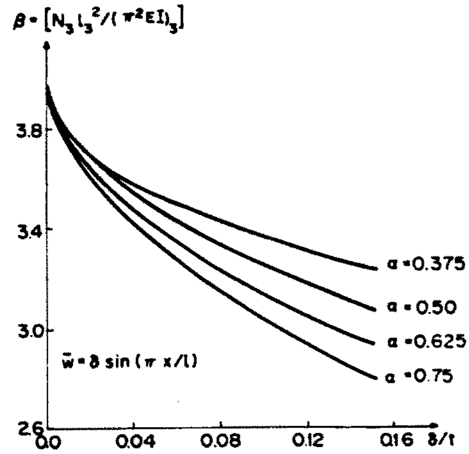

Fig. 4. Reduction of buckling load with increasing initial imperfection for different values of delamination ratio (α).

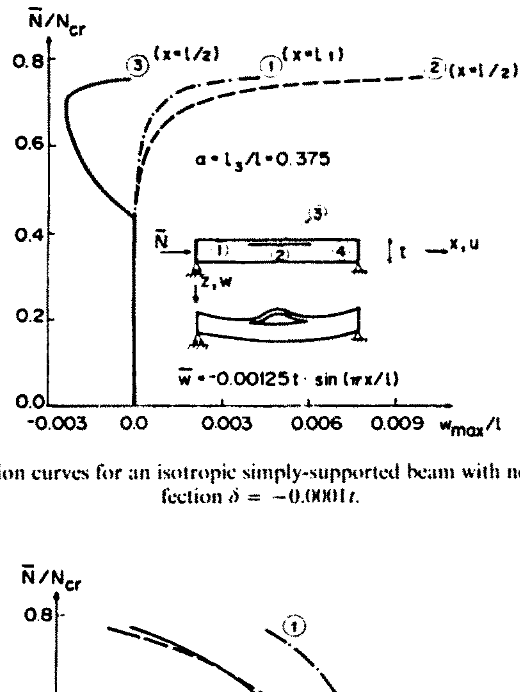

Fig. 5. Load deflection curves for an isotropic simply-supported beam with negative initial imperfection $\delta = -0.0001t$.

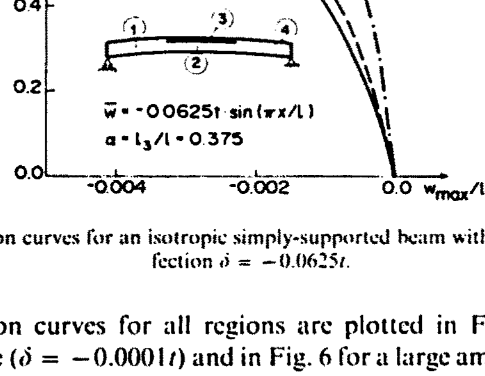

Fig. 6. Load deflection curves for an isotropic simply-supported beam with negative initial imperfection $\delta = -0.0625t$.

The load-deflection curves for all regions are plotted in Fig. 5 for a small initial imperfection amplitude ($\delta = -0.0001t$) and in Fig. 6 for a large amplitude ($\delta = -0.0625t$). It is seen (Fig. 5) that although the initial imperfection was upward, the buckling shape of the upper layer is upward but the beam deflects downward. For the larger amplitude (Fig.

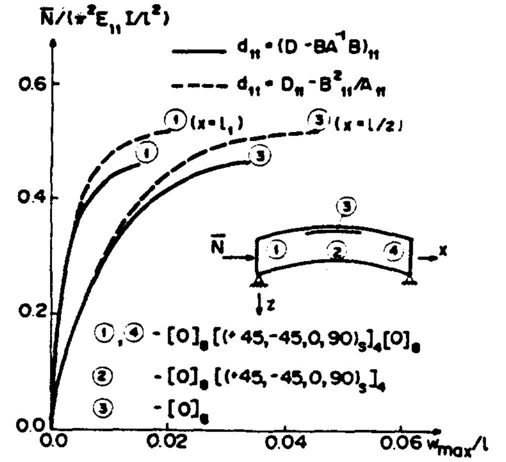

Fig. 7. Load-deflection curves of an anisotropic simply-supported beam with $B_{11}^{*} < 0$.

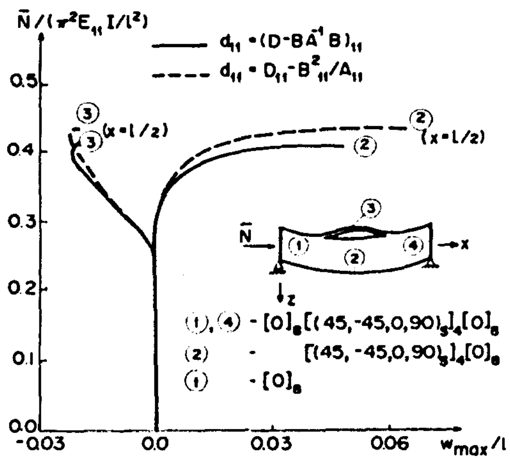

Fig. 8. Load deflection curves of an anisotropic simply-supported beam with $B_{11}^{*} > 0$.

6) no local buckling mode is distinguished, and the beam deflects upward. This is a very important observation, implying that the delaminated beam is sensitive to the type of initial imperfection. These two cases of a small and a large amplitude were also run with the aid of the NASTRAN code (MacNeal, 1986) and results were in very good agreement, but involved a much greater computational effort.

(b) *Anisotropic beam*

This example, which concerns a graphite/epoxy delaminated beam, is taken from Sheinman *et al.* (1989). The data for this example are: length $l = 60$ mm, delamination ratio $\alpha = l_3/l = 0.5$, 48 laminate with $h_{ply} = 0.125$ mm, thickness of laminate $= 6$ mm, thickness of delaminated upper layer $h = 1$ mm ($H = 5$ mm), $E_{11} = 1.3357 \times 10^{11}$ N m^{-2}, $E_{22} = 0.928 \times 10^{10}$ N m^{-2}, $G_{12} = 0.5765 \times 10^{10}$ N m^{-2} and $\nu_{12} = 0.342$. The symmetric stacking combination of $[0]_8[(45, -45, 0, 90)_s]_4[0]_8$ was chosen. Because of the delamination, the stacking of region 2 (which is $[0]_8[(45, -45, 0, 90)_s]_4$) is asymmetric and bending–stretching coupling is present. The initial imperfection is again $w(x) = \delta \sin(\pi x/l)$. (It should be mentioned that the stretching–bending coupling effect represents a built-in imperfection.) The buckling load of this case is given in Sheinman *et al.* (1989); here, only the post-buckling behavior is considered.

The stretching–bending coupling effect is examined in terms of the B_{11} parameter of region 2. The load–deflection for $B_{11} < 0$ (stacking $[0]_8[(45, -45, 0, 90)_s]_4[0]_8$) for $B_{11} > 0$ (stacking $[(45, -45, 0, 90)_s]_4[0]_8$) and for $B_{11} = 0$ (set artificially), are plotted in Figs 7, 8 and 9, respectively. It is seen that while at $B_{11} < 0$ no local buckling occurred, it did occur

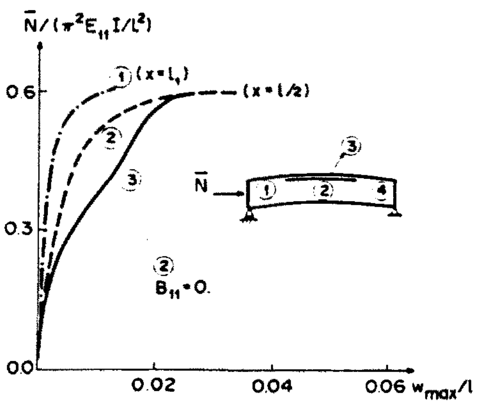

Fig. 9. Load-deflection curves of an anisotropic simply-supported beam with $B_{11}^* = 0$.

at $B_{11} > 0$ and the non-linear behavior is completely different. This can be explained by the fact that in the first case, region 2 tends to deflect upward and in the second it deflects downward. The effect of cylindrical bending (see Sheinman, 1989) was also checked by adopting the stiffness parameter $d_{11} = (D-BA^{-1}B)_{11}$ instead of $d_{11} = D_{11}-B_{11}^2/A_{11}$. It seems that consideration of cylindrical bending yields more flexible behavior, as illustrated by the solid lines in Figs 7 and 8. Finally, unlike the bifurcation point, the total load capacity is always smaller when $B_{11} \neq 0$, irrespective of whether it is positive or negative.

4. CONCLUSION

A non-linear analysis and a solution procedure are presented for delaminated beams of arbitrary stacking combination and boundary conditions under any external loading. The non-linear equations, which are based on the Von Karman kinematic approach, are solved by the modified Newton method and a finite difference scheme. The theory and solution procedure are general and suitable for investigating the effect of delamination on the overall non-linear behavior, as well as that of the initial imperfection. Of the principal findings, the following should be emphasized:

1. For a delaminated beam, the bifurcation point is only an indication of the overall behavior, and post-buckling analysis is called for.
2. The delaminated beam under axial compression is sensitive to the initial imperfection.
3. The stretching-bending coupling effect significantly reduces the stiffness of the beam.
4. Under cylindrical bending the beam behaves in a more flexible way.

Acknowledgements—The authors are indebted to Ing. E. Goldberg for editorial assistance.

REFERENCES

Chai, H., Babcock, C. D. and Krauss, W. G. (1981). One-dimensional modeling of failure in laminated plates by delamination buckling. *Int. J. Solids Struct.* **17**, 1069–1083.

MacNeal, R. H. (1986). Mac/Nastran Macneal Schwendler Corp., Los Angeles, Vol. 65.

Sheinman, I. (1989) On the cylindrical buckling load of laminated columns. *J. Engng Mech. ASCE* **115**(3), 659–661.

Sheinman, I. and Adan, M. (1987). The effect of shear deformation on post-buckling behavior of laminated beams. *ASME, J. Appl. Mech.* **54**, 558–562.

Sheinman, I. and Simitses, G. S. (1984). A modification of Potter's method for diagonal matrices with common unknown. *Comput. Struct.* **18**(2), 273–275.

Sheinman, I., Bass, M. and Ishai, O. (1989). Effect of delamination on stability of laminated composite strip. *Compos. Struct.* **11**, 227–242.

Shirakumar, K. N. and Whicomb, J. D. (1985). Buckling of sublaminate in a quasi-isotropic composite laminate. *J. Compos. Mater.* **19**, 2–18.

Simitses, G. D., Sallam, S. and Yin, W. L. (1985). Effect of delamination of axially loaded homogeneous laminated plates. *AIAA J.* **23**(9), 1437–1444.

Whitney, J. M. (1987). *Structural Analysis of Laminated Anisotropic Plates*. Technomic Publishing Co., Lancaster.

Yin, W. L. (1986). Cylindrical buckling of laminated and delaminated plates. *Proc. 27th AIAA/ASME/ASCE/AHS SDM Conf.*, Part 1, San Antonio, TX, pp. 165–179.

Yin, W. L., Sallam, S. and Simitses, G. J. (1984). Ultimate axial load capacity of delaminated plate. *Proc. 25th AIAA/ASME/ASCE/AHS SDM Conf.*, Paper 84-08-92, Palm Springs, CA.